

# Control of ordinary differential equations using Bagarello's operator approach : Case of forced harmonic oscillator systems

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**ABSTRACT.** This work deals with the study of an optimal control of a system of nonlinear differential equations using the Bagarello's operator approach recently introduced in a paper (*Int. Jour. of Theoretical Physics*, **43**, issue 12 (2004), p. 2371 - 2394). The control problem is reduced, by using the Pontryagin's maximum principle, to a system of ordinary differential equations with unknown state and adjoint variables. Its solution is then described in terms of a series expansion of commutators involving an unbounded self-adjoint, densely defined, system Hamiltonian operator  $H$  and initial position operators. Relevant simple applications are discussed.

**Keywords.** Optimal control; Pontryagin's maximum principle; ordinary differential equations; Hamiltonian operator; nonrelativistic quantum mechanics.

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## 1. Introduction

Various methods are used to solve systems of ordinary differential equations (SODEs) in mathematics and applied sciences. The most popular ones include factorization [15], linearization [17], perturbation [14, 20], closure approximation [1, 2], discretization [23] and Adomian methods [1, 3, 8, 9, 16] to cite a few. Recently, in an interesting paper [7], Bagarello developed a non-commutative method, based on the quantum mechanics formalism, for the analysis of systems of ODEs. He provided solutions of some systems described by unbounded self-adjoint and densely defined Hamiltonian operators and discussed corresponding integrals of motion.

The possibility of acting, in an appropriate way, on systems of ODEs governing physical phenomena or other mechanisms in nature gives rise to the control theory. Optimal control problems can be solved by the so-called Pontryagin's method.

One of the most important systems encountered in the literature is certainly the system of harmonic oscillators, widely used as a basic tool in physics [12, 19]. This is motivated by their role in many applications in various fields of physics and technology. The harmonic oscillators describe a wide class of physical models. Besides, their properties are well known. This explains

why they are often used as a preliminary tool in order to gain initial insight into complex systems. For instance, quantum mechanics as well as optimal control were first illustrated using systems of harmonic oscillators. A typical controlled harmonic oscillator is a pendulum made of a string and a ball moving in a vertical plane and subjected to a force (also called the control) whose role is to stress the system to rest in a minimum time.

Recently, Andresen et al [4] investigated the control of an oscillator using the frequency variation. Dasanayake [11] and Van Dooren [25], (see also references therein), performed numerical solutions of optimal control problems by pseudospectral methods and Chebyshev series.

In the present work, for the first time to the best of our knowledge, the Bagarello's noncommutative approach is applied to the control of systems of nonlinear ordinary differential equations.

The paper is organized as follows. In Section 2, we present the considered optimal control problem. We derive the state and adjoint systems of ordinary differential equations using the Pontryagin's maximum principle [13, 21]. Then, we use the Bagarello's operator method to solve such systems. In Section 3, harmonic oscillator systems are controlled and discussed.

## 2. Theoretical framework

We consider the following optimal control problem [6, 24]:

$$\dot{x} = f(x, u), \quad x(0) = x_0 \quad (1)$$

$$x(T) = x_1 \quad (2)$$

$$u \in V \subset E^r \quad (3)$$

$$\min_{u \in V} \int_0^T f^0(x, u) dt, \quad (4)$$

where  $x, x_0, x_1 \in E^n$ ;  $E^r \supset V \ni u$  is the control,  $V$  the control set;  $T$  stands for the moment and  $E$  is the Euclidean space. All the functions occurring in the formulation of the problem are assumed to be differentiable with respect to  $(x, u)$ . Further, the minimum should be in a class of measurable and bounded functions  $u$ , defined on  $[0, T]$ . For instance,  $V$  can be represented in the form

$$V = \{u \in E^r : h_i(u) \leq 0, i \in I_1; h_i(u) = 0, i \in I_2\}$$

where the functions  $h_i$  express the constraints on the control;  $I_1$  and  $I_2$  are finite disjoint index sets.

Let us introduce the Hamilton-Pontryagin's function

$$\eta(x, u, \psi) = \psi f^*(x, u) - \psi^0 f^0(x, u)$$

where  $f = (f_1, \dots, f_n) \in E^n$ ;  $E^n \ni \psi = (\psi_1, \dots, \psi_n)$  is the adjoint variable depending on  $t$ ;  $*$  stands for the transposition operation.  $\psi^0$  is a positive number.

By virtue of the Pontryagin's maximum principle [13, 21], the existence of adjoint functions is a necessary condition to link the control system of ordinary differential equations to a functional objective. The optimal control  $u^*$  is then characterized in terms of the state and adjoint functions.

Thus, given an optimal control  $u^*$  and the corresponding system (1), there exists an adjoint variable  $\psi$  satisfying the following equation:

$$\frac{d\psi}{dt} = -\frac{\partial \eta}{\partial x}(x, u^*(x, \psi), \psi). \quad (5)$$

There results the Cauchy problem

$$\dot{x} = f(x, u^*(x, \psi)), \quad x(0) = x_0 \quad (6)$$

$$\dot{\psi} = -\frac{\partial \eta}{\partial x}(x, u^*(x, \psi), \psi), \quad \psi(0) = \psi_0 \quad (7)$$

with the unknowns  $x, \psi$ .

$$\text{Setting } y = \begin{pmatrix} x \\ \psi \end{pmatrix}, g(y) = \begin{pmatrix} f(x, u^*(x, \psi)) \\ -\frac{\partial \eta}{\partial x}(x, u^*(x, \psi), \psi) \end{pmatrix}, y(0) = \begin{pmatrix} x(0) \\ \psi(0) \end{pmatrix},$$

we can write the latter in the form:

$$\dot{y} = g(y), \quad y(0) = y_0 = \begin{pmatrix} x_0 \\ \psi_0 \end{pmatrix}, \quad (8)$$

or, explicitly,

$$\begin{cases} \dot{y}_1 &= g_1(y_1, y_2, \dots, y_{2n}) \\ \dot{y}_2 &= g_2(y_1, y_2, \dots, y_{2n}) \\ \vdots &= \vdots \\ \dot{y}_{2n} &= g_{2n}(y_1, y_2, \dots, y_{2n}) \\ y_j(0) &= y_j^0, \quad j = 1, \dots, 2n. \end{cases} \quad (9)$$

We suppose that the functions  $g_j$  are such that the solution of the Cauchy problem (9) exists and is unique.

In this work, we intend to solve this system by Bagarello's approach ([7], and references therein), based on a quantum mechanical formalism. Given a quantum mechanical system  $\mathcal{S}$  and the related set of observables  $\mathcal{O}_{\mathcal{S}}$ , i. e., the set of all self-adjoint bounded (or more often unbounded) operators describing  $\mathcal{S}$ , the evolution of any observable  $Y \in \mathcal{O}_{\mathcal{S}}$  satisfies the Heisenberg equation of motion (HOEM)[7]:

$$\frac{d}{dt}Y(t) = i[H, Y(t)]. \quad (10)$$

Here  $[A, B] = AB - BA$  is the commutator between  $A, B \in \mathcal{O}_{\mathcal{S}}$ ;  $H$  is assumed to be a densely defined self-adjoint Hamiltonian operator of the system acting on some Hilbert space  $\mathcal{H}$ , given by [7]

$$H(\vec{g}_0) = \frac{1}{2} \sum_{j=1}^{2n} \{p_j g_j(y_1^0, y_2^0, \dots, y_{2n}^0) + g_j(y_1^0, y_2^0, \dots, y_{2n}^0) p_j\} \quad (11)$$

where  $\vec{g}_0 = (g_1(\vec{Y}^0), g_2(\vec{Y}^0), \dots, g_{2n}(\vec{Y}^0))$ ,  $\vec{Y}^0 = (y_1^0, y_2^0, \dots, y_{2n}^0)$ ; the initial position  $y_j^0$  is considered as an operator acting on the Hilbert space  $\mathcal{H}$ , and  $p_j$  is its canonical conjugate momentum operator such that

$$[y_j^0, p_k] = i\delta_{j,k}\mathbf{I}, \quad j, k = 1, \dots, 2n \quad (12)$$

$$[y_j^0, y_k^0] = [p_j, p_k] = 0, \quad j, k = 1, \dots, 2n. \quad (13)$$

Standard results in quantum mechanics show that for any differentiable functions

$\varphi(y_1^0, y_2^0, \dots, y_{2n}^0)$ , and  $\hat{\varphi}(p_1, p_2, \dots, p_{2n})$ , we have

$$[p_j, \varphi(y_1^0, y_2^0, \dots, y_{2n}^0)] = -i \frac{\partial \varphi}{\partial y_j^0}, \quad j = 1, \dots, 2n \quad (14)$$

$$[y_j^0, \hat{\varphi}(p_1, p_2, \dots, p_{2n})] = i \frac{\partial \hat{\varphi}}{\partial p_j}, \quad j = 1, \dots, 2n. \quad (15)$$

**THEOREM 1.** [7] *If the functions  $g_j$  are holomorphic, a formal solution of the HOEM (9) is*

$$Y(t) = e^{iHt} Y^0 e^{-iHt}. \quad (16)$$

where  $Y^0$  is the initial value of  $Y(t)$  and  $H$  does not depend explicitly on time.

Furthermore if  $H$  is bounded, we get

$$Y(t) = \sum_{k \geq 0} \frac{(it)^k}{k!} [H, Y^0]_k \quad (17)$$

where  $[A, B]_k$  is the multiple commutator recursively defined as :

$$[A, B]_0 = B; \quad [A, B]_k = [A, [A, B]_{k-1}].$$

The concepts of integral of motion and extended integral of motion of the system (9) are defined as in [7].

DEFINITION 1. Any holomorphic function  $I$  depending on the variables  $y_j$ , such that

$$I(y_1(t), y_2(t), \dots, y_{2n}(t)) = I_0 \quad \forall t, \quad I_0 = \text{const.},$$

is called an *Integral of motion (IoM)* of system (9).

DEFINITION 2. We call *extended integral of motion (EIoM)* of the system (9) any holomorphic function  $J$  depending on the  $y_j, p_j$  such that

$$J(y_1(t), y_2(t), \dots, y_{2n}(t), p_1(t), p_2(t), \dots, p_{2n}(t)) = J_0 \quad \forall t, J_0 = \text{const.},$$

where  $y_j(t) = e^{iHt} y_j^0 e^{-iHt}$  and  $p_j(t) = e^{iHt} p_j e^{-iHt}$ ,  $j = 1, \dots, 2n$ .

One of the main advantages of the Bagarello's strategy is to provide at hand a good approximation of the solution of the SODE. The  $N$ -th order approximation ( $N \in \mathbb{N}$ ) is

$$Y_N(t) = \sum_{k=0}^N \frac{(it)^k}{k!} [H, Y^0]_k. \quad (18)$$

Using the integral of motion, we can estimate the approximation's error of the unknown exact solution of the derived Cauchy problem. In this case, one can proceed as follows : let  $Y_N(t)$  be the approximated solution of  $Y(t)$ . We may thus compute and evaluate the error [7]

$$\Delta_N(t) = I(Y_N(t)) - I(Y^0). \quad (19)$$

This approach is useful for our study because formula (18) provides a simple approximation scheme for exact solutions to ordinary differential equations. Further the estimation of the error  $\Delta_N(t)$  tells us how good this approximation is.

In the next section, we apply this approach to time-optimal control problems.

### 3. Applications

**3.1. Problem 1.** In classical mechanics, we study the forced harmonic oscillator. The corresponding optimal control problem has the form [18]

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\omega^2 x_1 + u(t) \\ x_1(0) &= x_1^0; \quad x_2(0) = x_2^0; \\ x_1(T) &= 0; \quad x_2(T) = 0; \\ -1 &\leq u \leq 1 \\ T &\longrightarrow \inf. \end{cases} \quad (20)$$

Here  $t$  and  $T$  denote the time,  $x = (x_1, x_2) \in E^2$ ,  $u$  is the control and  $\omega$  the oscillation frequency.

In this problem, the infimum is sought in a class of controls  $u(t)$ ,  $t \geq 0$ . We study a control  $u$  that moves the point  $x(0)$  to the point  $x(T) = 0$  in accordance with the corresponding solution of (20) during the time  $T$ . Then we solve the boundary value problem derived from the Pontryagin's maximum principle.

Let us consider the Hamilton-Pontryagin's function

$$\eta(x, u, \psi) = -1 + \psi_1 x_2 + \psi_2 (-\omega^2 x_1 + u) \quad (21)$$

where  $\psi = (\psi_1, \psi_2) \in E^2$ .

According to Pontryagin's maximum principle, we obtain the control  $u$  in the form

$$u(t) = \begin{cases} 1 & \text{if } \psi_2(t) > 0 \\ -1 & \text{if } \psi_2(t) < 0. \end{cases}$$

Now we solve the Cauchy problem derived from the Pontryagin's maximum principle using the technique developed by Bagarello.

In the case  $u(t) = 1$ , we can write

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\omega^2 x_1 + 1 \\ \dot{\psi}_1 = \omega^2 \psi_2 \\ \dot{\psi}_2 = -\psi_1 \end{cases} \quad (22)$$

$$x_1(0) = x_1^0; \quad x_2(0) = x_2^0; \quad \psi_1(0) = \psi_1^0; \quad \psi_2(0) = \psi_2^0.$$

Introducing the new variable  $\tilde{x}_1 = x_1 - \frac{1}{\omega^2}$ , we rewrite (22) in the form

$$\begin{cases} \dot{\tilde{x}}_1 = x_2 \\ \dot{x}_2 = -\omega^2 \tilde{x}_1 \\ \dot{\psi}_1 = \omega^2 \psi_2 \\ \dot{\psi}_2 = -\psi_1 \end{cases} \quad (23)$$

In the case  $u(t) = -1$  we get

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\omega^2 x_1 - 1 \\ \dot{\psi}_1 = \omega^2 \psi_2 \\ \dot{\psi}_2 = -\psi_1 \end{cases} \quad (24)$$

or, equivalently, denoting  $\hat{x}_1 = x_1 + \frac{1}{\omega^2}$ ,

$$\begin{cases} \dot{\hat{x}}_1 = x_2 \\ \dot{x}_2 = -\omega^2 \hat{x}_1 \\ \dot{\psi}_1 = \omega^2 \psi_2 \\ \dot{\psi}_2 = -\psi_1 \end{cases} \quad (25)$$

with the previous initial conditions .

Equations (23) and (25) reduce to the equivalent system

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -\omega^2 y_1 \\ \dot{y}_3 = \omega^2 y_4 \\ \dot{y}_4 = -y_3 \end{cases} \quad (26)$$

where  $y_1 = \tilde{x}_1$  or  $y_1 = \hat{x}_1, y_2 = x_2, y_3 = \psi_1, y_4 = \psi_2$ .

Here, instead of using standard methods widely spread in the literature, we show that Bagarello's suggestion can also be exploited to solve this optimal control problem and to reproduce known results. Indeed, using the definition of the commutator

$$g_j p_j = -[p_j, g_j] + p_j g_j, \quad j = 1, \dots, 4,$$

we compute the Hamiltonian of the previous system in the form

$$H = p_1 y_2^0 - \omega^2 p_2 y_1^0 + \omega^2 p_3 y_4^0 - p_4 y_3^0. \quad (27)$$

Developing the solution of (26) as an infinite series:

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{pmatrix} = \begin{pmatrix} y_1^0 \\ y_2^0 \\ y_3^0 \\ y_4^0 \end{pmatrix} + it \left[ H, \begin{pmatrix} y_1^0 \\ y_2^0 \\ y_3^0 \\ y_4^0 \end{pmatrix} \right] + \frac{(it)^2}{2!} \left[ H, \begin{pmatrix} y_1^0 \\ y_2^0 \\ y_3^0 \\ y_4^0 \end{pmatrix} \right]_2 + \frac{(it)^3}{3!} \left[ H, \begin{pmatrix} y_1^0 \\ y_2^0 \\ y_3^0 \\ y_4^0 \end{pmatrix} \right]_3 + \dots$$

we calculate the multiple commutator

$$\begin{pmatrix} [H, y_1^0]_{2p} \\ [H, y_2^0]_{2p} \\ [H, y_3^0]_{2p} \\ [H, y_4^0]_{2p} \end{pmatrix} = \begin{pmatrix} \omega^{2p} y_1^0 \\ \omega^{2p} y_2^0 \\ \omega^{2p} y_3^0 \\ \omega^{2p} y_4^0 \end{pmatrix}; \quad \begin{pmatrix} [H, y_1^0]_{2p+1} \\ [H, y_2^0]_{2p+1} \\ [H, y_3^0]_{2p+1} \\ [H, y_4^0]_{2p+1} \end{pmatrix} = \begin{pmatrix} -i\omega^{2p} y_2^0 \\ i\omega^{2p+2} y_1^0 \\ -i\omega^{2p+2} y_4^0 \\ i\omega^{2p} y_3^0 \end{pmatrix}, p = 0, 1, 2, 3, \dots$$

and obtain

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{pmatrix} = \begin{pmatrix} y_1^0 \cos(\omega t) + \frac{y_2^0}{\omega} \sin(\omega t) \\ y_2^0 \cos(\omega t) - y_1^0 \omega \sin(\omega t) \\ y_3^0 \cos(\omega t) + y_4^0 \omega \sin(\omega t) \\ y_4^0 \cos(\omega t) - \frac{y_3^0}{\omega} \sin(\omega t) \end{pmatrix}.$$

In order to determine the time  $T$  such that

$$\begin{cases} x_1(T) = 0 \\ x_2(T) = 0 \end{cases},$$

and taking into account  $y_1 = \tilde{x}_1 = x_1 - \frac{1}{\omega^2}$ ,  $y_2 = x_2$ , we obtain

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} (x_1^0 - \frac{1}{\omega^2}) \cos(\omega t) + \frac{x_2^0}{\omega} \sin(\omega t) + \frac{1}{\omega^2} \\ x_2^0 \cos(\omega t) - (x_1^0 - \frac{1}{\omega^2}) \omega \sin(\omega t) \end{pmatrix}$$

and

$$\cos(\omega T) = \frac{\omega}{x_2^0} \left( x_1^0 - \frac{1}{\omega^2} \right) \sin(\omega T); \quad \sin(\omega T) = \frac{-x_2^0/\omega}{\omega^2 (x_1^0 - \frac{1}{\omega^2})^2 + (x_2^0)^2}.$$

Thus

$$\tan \omega T = \frac{x_2^0}{\omega(x_1^0 - \frac{1}{\omega^2})} \quad \text{or} \quad \omega T = \arctan \frac{x_2^0}{\omega(x_1^0 - \frac{1}{\omega^2})};$$

$$\tilde{T} = \frac{1}{\omega} \arctan \frac{x_2^0}{\omega(x_1^0 - \frac{1}{\omega^2})}, \quad x_2^0 \left( x_1^0 - \frac{1}{\omega^2} \right) > 0$$

Analogously, for  $y_1 = \hat{x}_1 = x_1 + \frac{1}{\omega^2}$ ,  $y_2 = x_2$ , we get

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} (x_1^0 + \frac{1}{\omega^2}) \cos(\omega t) + \frac{x_2^0}{\omega} \sin(\omega t) - \frac{1}{\omega^2} \\ x_2^0 \cos(\omega t) - (x_1^0 + \frac{1}{\omega^2}) \omega \sin(\omega t) \end{pmatrix}$$

and the relations

$$\cos(\omega T) = \frac{\omega}{x_2^0} \left( x_1^0 + \frac{1}{\omega^2} \right) \sin(\omega T); \quad \sin(\omega T) = \frac{x_2^0/\omega}{\omega^2 (x_1^0 + \frac{1}{\omega^2})^2 + (x_2^0)^2}$$

implying

$$\tan \omega T = \frac{x_2^0}{\omega(x_1^0 + \frac{1}{\omega^2})} \quad \text{or} \quad \omega T = \arctan \frac{x_2^0}{\omega(x_1^0 + \frac{1}{\omega^2})};$$

$$\hat{T} = \frac{1}{\omega} \arctan \frac{x_2^0}{\omega(x_1^0 + \frac{1}{\omega^2})}, \quad x_2^0 \left( x_1^0 + \frac{1}{\omega^2} \right) > 0.$$

Therefore, the optimal time solution to problem (20) is given by considering the following two cases : if  $x_2^0 > 0$ , then  $\hat{T}$  is the optimal time and if  $x_2^0 < 0$ , then  $\tilde{T}$  is the optimal time.

In the sequel, we deal with more complicated problems whose solutions can be obtained by perturbative approaches. In this case, we show that as a possible candidate, Bagarello's formalism can be easily implemented in a suitable and solvable form.

**3.2. Problem 2.** Let us consider the optimal control problem of a pendulum with large oscillations in the form [26]

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\beta x_2 - \sin x_1 + u(t) \\ x_1(0) &= x_1^0; \quad x_2(0) = x_2^0; \\ x_1(T) &= 0; \quad x_2(T) = 0; \\ -1 &\leq u \leq 1 \\ T &\rightarrow \inf \end{cases} \quad (28)$$

$u(t) = \frac{1}{m}F(t)$ ,  $0 \leq t \leq T$ , is the control subjected to the following constraint :

$$u \in V = \{u \in E^1 : |u| \leq 1\}; \quad (29)$$

$x_1^0, x_2^0$  are given positive constants,  $m$  is the mass,  $F$  the force and  $\beta > 0$  a constant.

The Hamilton-Pontryagin's function is written as

$$\eta(x, u, \psi) = -1 + \psi_1 x_2 + \psi_2 (-\beta x_2 - \sin x_1 + u) \quad (30)$$

where  $\psi = (\psi_1, \psi_2) \in E^2$ .

By virtue of the Pontryagin's maximum principle, the control is given by

$$u(t) = \begin{cases} 1 & \text{if } \psi_2(t) > 0 \\ -1 & \text{if } \psi_2(t) < 0. \end{cases}$$

The associated Cauchy problem reads

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\beta x_2 - \sin x_1 + u(t) \\ \dot{\psi}_1 = \psi_2 \cos x_1 \\ \dot{\psi}_2 = -\psi_1 + \beta \psi_2 \end{cases} \quad (31)$$

with the initial conditions

$$x_1(0) = x_1^0, x_1^0 > 0; \quad x_2(0) = x_2^0, x_2^0 > 0; \quad \psi_1(0) = \psi_1^0; \quad \psi_2(0) = \psi_2^0.$$

Equivalently, this can be re-expressed as follows:

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -\beta y_2 - \sin y_1 + u(t) \\ \dot{y}_3 = y_4 \cos y_1 \\ \dot{y}_4 = -y_3 + \beta y_4 \end{cases} \quad (32)$$

$$y_1(0) = y_1^0; \quad y_2(0) = y_2^0; \quad y_3(0) = y_3^0; \quad y_4(0) = y_4^0$$

where  $y_1 = x_1, y_2 = x_2, y_3 = \psi_1, y_4 = \psi_2$ .

The Hamiltonian of the system takes the form:

$$H = p_1 y_2^0 + p_2 [-\beta y_2^0 - \sin y_1^0 + u(t)] + p_3 y_4^0 \cos y_1^0 + p_4 (-y_3^0 + \beta y_4^0). \quad (33)$$

The solution (18) becomes in this case:

$$Y_2(t) = \begin{pmatrix} y_1^0 \\ y_2^0 \\ y_3^0 \\ y_4^0 \end{pmatrix} + it \left[ H, \begin{pmatrix} y_1^0 \\ y_2^0 \\ y_3^0 \\ y_4^0 \end{pmatrix} \right] + \frac{(it)^2}{2!} \left[ H, \begin{pmatrix} y_1^0 \\ y_2^0 \\ y_3^0 \\ y_4^0 \end{pmatrix} \right]_2$$

where the commutators are given by

$$\begin{pmatrix} [H, y_1^0] \\ [H, y_2^0] \\ [H, y_3^0] \\ [H, y_4^0] \end{pmatrix} = \begin{pmatrix} -iy_2^0 \\ -i[-\beta y_2^0 - \sin y_1^0 + u(t)] \\ -iy_4^0 \cos y_1^0 \\ -i(-y_3^0 + \beta y_4^0) \end{pmatrix};$$

$$\begin{pmatrix} [H, y_1^0]_2 \\ [H, y_2^0]_2 \\ [H, y_3^0]_2 \\ [H, y_4^0]_2 \end{pmatrix} = \begin{pmatrix} (-i)^2 [-\beta y_2^0 - \sin y_1^0 + u(t)] \\ (-i)^2 \{-\beta [-\beta y_2^0 - \sin y_1^0 + u(t)] - y_2^0 \cos(y_1^0)\} \\ (-i)^2 [-y_2^0 y_4^0 \sin(y_1^0) + (-y_3^0 + \beta y_4^0) \cos(y_1^0)] \\ (-i)^2 [-y_4^0 \cos(y_1^0) + \beta(-y_3^0 + \beta y_4^0)] \end{pmatrix}.$$

Finally we get

- For  $u(t) = -1$ ,

$$\begin{pmatrix} \tilde{y}_{1.2}(t) \\ \tilde{y}_{2.2}(t) \end{pmatrix} = \begin{pmatrix} y_1^0 + t y_2^0 + \frac{t^2}{2} (-\beta y_2^0 - \sin y_1^0 - 1) \\ y_2^0 + t(-\beta y_2^0 - \sin y_1^0 - 1) + \frac{t^2}{2} [-\beta(-\beta y_2^0 - \sin y_1^0 - 1) - y_2^0 \cos y_1^0] \end{pmatrix}$$

and

$$\begin{pmatrix} \tilde{x}_{1.2}(t) \\ \tilde{x}_{2.2}(t) \end{pmatrix} = \begin{pmatrix} x_1^0 + t x_2^0 + \frac{t^2}{2} (-\beta x_2^0 - \sin x_1^0 - 1) \\ x_2^0 + t(-\beta x_2^0 - \sin x_1^0 - 1) + \frac{t^2}{2} [-\beta(-\beta x_2^0 - \sin x_1^0 - 1) - x_2^0 \cos x_1^0] \end{pmatrix}$$

- For  $u(t) = 1$ ,

$$\begin{pmatrix} \hat{y}_{1.2}(t) \\ \hat{y}_{2.2}(t) \end{pmatrix} = \begin{pmatrix} y_1^0 + t y_2^0 + \frac{t^2}{2} (-\beta y_2^0 - \sin y_1^0 + 1) \\ y_2^0 + t(-\beta y_2^0 - \sin y_1^0 + 1) + \frac{t^2}{2} [-\beta(-\beta y_2^0 - \sin y_1^0 + 1) - y_2^0 \cos y_1^0] \end{pmatrix}$$

and

$$\begin{pmatrix} \hat{x}_{1.2}(t) \\ \hat{x}_{2.2}(t) \end{pmatrix} = \begin{pmatrix} x_1^0 + t x_2^0 + \frac{t^2}{2} (-\beta x_2^0 - \sin x_1^0 + 1) \\ x_2^0 + t(-\beta x_2^0 - \sin x_1^0 + 1) + \frac{t^2}{2} [-\beta(-\beta x_2^0 - \sin x_1^0 + 1) - x_2^0 \cos x_1^0] \end{pmatrix}.$$

Taking into account the relations

$$\begin{cases} \tilde{x}_{1.2}(T) = 0 \\ \tilde{x}_{2.2}(T) = 0 \end{cases} \quad \text{and} \quad \begin{cases} \hat{x}_{1.2}(T) = 0 \\ \hat{x}_{2.2}(T) = 0 \end{cases},$$

we obtain

$$(i) \quad \tilde{T} = \frac{(-\beta x_2^0 - \sin x_1^0 - 1)(x_2^0 + \beta x_1^0) + x_1^0 x_2^0 \cos x_1^0}{(-\beta x_2^0 - \sin x_1^0 - 1)(\sin x_1^0 + 1) - (x_2^0)^2 \cos x_1^0}$$

$$\text{and } a_2 [(a_1 c_2 - a_2 c_1)^2 + (a_2 b_1 - a_1 b_2)(b_1 c_2 - b_2 c_1)] = 0$$

with  $a_1, b_1, c_1$  and  $a_2, b_2, c_2$  being respectively the coefficients of the polynomials  $\tilde{x}_{1.2}(t)$  and  $\tilde{x}_{2.2}(t)$ .

$$(ii) \quad \hat{T} = \frac{(-\beta x_2^0 - \sin x_1^0 + 1)(x_2^0 + \beta x_1^0) + x_1^0 x_2^0 \cos x_1^0}{(-\beta x_2^0 - \sin x_1^0 + 1)(\sin x_1^0 - 1) - (x_2^0)^2 \cos x_1^0}$$

$$\text{and } A_2 [(A_1 C_2 - A_2 C_1)^2 + (A_2 B_1 - A_1 B_2)(B_1 C_2 - B_2 C_1)] = 0$$

with  $A_1, B_1, C_1$  and  $A_2, B_2, C_2$  being respectively the coefficients of the polynomials  $\hat{x}_{1.2}(t)$  and  $\hat{x}_{2.2}(t)$ .

$$\text{If } \frac{(-\beta x_2^0 - \sin x_1^0 - 1)(x_2^0 + \beta x_1^0) + x_1^0 x_2^0 \cos x_1^0}{(-\beta x_2^0 - \sin x_1^0 - 1)(\sin x_1^0 + 1) - (x_2^0)^2 \cos x_1^0} <$$

$$\frac{(-\beta x_2^0 - \sin x_1^0 + 1)(x_2^0 + \beta x_1^0) + x_1^0 x_2^0 \cos x_1^0}{(-\beta x_2^0 - \sin x_1^0 + 1)(\sin x_1^0 - 1) - (x_2^0)^2 \cos x_1^0}$$

then  $\tilde{T}$  is the solution of the problem. Otherwise  $\hat{T}$  is the solution.

Let us estimate the error using the following integral of motion

$$I(x_1, x_2) = x_2 + \beta x_1 + \int_0^t [\sin x_1(\tau) - u(\tau)] d\tau; \quad \text{with } I(x_1^0, x_2^0) = x_2^0 + \beta x_1^0.$$

Then



(i) for  $u(t) = -1$ ,

$$\begin{aligned}\tilde{\Delta}_2(t) &= I(\tilde{x}_{1.2}, \tilde{x}_{2.2}) - I(x_1^0, x_2^0) = t(-\sin x_1^0 - 1) - \frac{t^2}{2}x_2^0 \cos x_1^0 \\ &\quad + \int_0^t \left\{ \sin \left[ x_1^0 + \tau x_2^0 + \frac{\tau^2}{2}(-\beta x_2^0 - \sin x_1^0 - 1) \right] + 1 \right\} d\tau; \\ |\tilde{\Delta}_2(t)| &\leq \frac{t^2}{2}x_2^0 + 4t\end{aligned}$$

leading to  $|\tilde{\Delta}_2(t)| < \kappa$  for  $t \in \left[0, \frac{-4 + \sqrt{16 + 2x_2^0 \kappa}}{x_2^0}\right]$ , with  $\kappa$  a sufficiently small positive number.

(ii) for  $u(t) = 1$ ,

$$\begin{aligned}\hat{\Delta}_2(t) &= I(\hat{x}_{1.2}, \hat{x}_{2.2}) - I(x_1^0, x_2^0) = t(-\sin x_1^0 + 1) - \frac{t^2}{2}x_2^0 \cos x_1^0 \\ &\quad + \int_0^t \left\{ \sin \left[ x_1^0 + \tau x_2^0 + \frac{\tau^2}{2}(-\beta x_2^0 - \sin x_1^0 + 1) \right] - 1 \right\} d\tau; \\ |\hat{\Delta}_2(t)| &\leq \frac{t^2}{2}x_2^0 + 4t,\end{aligned}$$

yielding  $|\hat{\Delta}_2(t)| < \kappa$  for  $t \in \left[0, \frac{-4 + \sqrt{16 + 2x_2^0 \kappa}}{x_2^0}\right]$ , with  $\kappa$  a sufficiently small positive number.

**3.3. Problem 3.** Let us examine the optimal control problem when the state equation is the Van der Pol equation [5, 10]

$$\ddot{x} + \varepsilon \dot{x}(x^2 - 1) + x = u(t) \quad (34)$$

such that the control  $u(t) \in [\alpha, \beta]$ . The problem has the form

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \varepsilon x_2(1 - x_1^2) + u(t) \\ x_1(0) &= x_1^0; \quad x_2(0) = x_2^0; \\ x_1(T) &= 0; \quad x_2(T) = 0; \\ \alpha &\leq u \leq \beta \\ T &\rightarrow \inf \end{cases} \quad (35)$$

where  $\varepsilon, \alpha, \beta$  are real constants.

The Hamilton-Pontryagin's function is

$$\eta(x, u, \psi) = -1 + \psi_1 x_2 + \psi_2 [-x_1 + \varepsilon x_2(1 - x_1^2) + u(t)]. \quad (36)$$

According to the Pontryagin's maximum principle, the supremum of the function  $\eta$  depending on  $x_1, x_2, \psi_1, \psi_2, u$  with respect to  $u$  is reached when the control takes the following form:

$$u(t) = \begin{cases} \beta & \text{if } \psi_2(t) > 0 \\ \alpha & \text{if } \psi_2(t) < 0 \end{cases}$$

Two cases are examined

(i) Taking  $u(t) = \alpha$ , we get the following Cauchy problem

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + \varepsilon x_2(1 - x_1^2) + \alpha \\ \dot{\psi}_1 = (1 + 2\varepsilon x_1 x_2)\psi_2 \\ \dot{\psi}_2 = -[\psi_1 + \varepsilon(1 - x_1^2)\psi_2] \\ x_1(0) = x_1^0; \quad x_2(0) = x_2^0; \quad \psi_1(0) = \psi_1^0; \quad \psi_2(0) = \psi_2^0, \end{cases} \quad (37)$$

which, according to (9), can be reduced to

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -y_1 + \varepsilon y_2(1 - y_1^2) + \alpha \\ \dot{y}_3 = (1 + 2\varepsilon y_1 y_2) y_4 \\ \dot{y}_4 = -[y_3 + \varepsilon(1 - y_1^2) y_4] \\ y_1(0) = y_1^0; \quad y_2(0) = y_2^0; \quad y_3(0) = y_3^0; \quad y_4(0) = y_4^0. \end{cases} \quad (38)$$

The Hamiltonian of the system is computed as follows

$$\begin{aligned} H = p_1 y_2^0 + p_2 \{ -y_1^0 + \varepsilon y_2^0 [1 - (y_1^0)^2] + \alpha \} + p_3 (1 + 2\varepsilon y_1^0 y_2^0) y_4^0 \\ + p_4 \{ y_3^0 + \varepsilon [1 - (y_1^0)^2] y_4^0 \} + i\varepsilon [1 - (y_1^0)^2]. \end{aligned} \quad (39)$$

The first order approximation solutions are given by

$$\begin{aligned} \begin{pmatrix} H, y_1^0 \\ H, y_2^0 \\ H, y_3^0 \\ H, y_4^0 \end{pmatrix} &= \begin{pmatrix} -iy_2^0 \\ -i \{ -y_1^0 + \varepsilon y_2^0 [1 - (y_1^0)^2] + \alpha \} \\ -i [(1 + 2\varepsilon y_1^0 y_2^0) y_4^0] \\ -i \{ y_3^0 + \varepsilon [1 - (y_1^0)^2] y_4^0 \} \end{pmatrix}; \\ \begin{pmatrix} \tilde{y}_{1.1}(t) \\ \tilde{y}_{2.1}(t) \end{pmatrix} &= \begin{pmatrix} y_1^0 + t y_2^0 \\ y_2^0 + t \{ -y_1^0 + \varepsilon y_2^0 [1 - (y_1^0)^2] + \alpha \} \end{pmatrix}. \end{aligned}$$

In the original variables

$$\begin{pmatrix} \tilde{x}_{1.1}(t) \\ \tilde{x}_{2.1}(t) \end{pmatrix} = \begin{pmatrix} x_1^0 + t x_2^0 \\ x_2^0 + t \{ -x_1^0 + \varepsilon x_2^0 [1 - (x_1^0)^2] + \alpha \} \end{pmatrix}.$$

(ii) For  $u(t) = \beta$ , the corresponding derived Cauchy problem is put in the form:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + \varepsilon x_2(1 - x_1^2) + \beta \\ \dot{\psi}_1 = (1 + 2\varepsilon x_1 x_2) \psi_2 \\ \dot{\psi}_2 = -[\psi_1 + \varepsilon(1 - x_1^2) \psi_2] \\ x_1(0) = x_1^0; \quad x_2(0) = x_2^0; \quad \psi_1(0) = \psi_1^0; \quad \psi_2(0) = \psi_2^0. \end{cases} \quad (40)$$

This SODE differs from (38) only by the term  $\beta$  replacing  $\alpha$ . Then, replacing *mutatis mutandis*  $\alpha$  by  $\beta$ , the SODE remains the same as in (38). Hence, the first order approximation solutions are given in the form:

$$\begin{pmatrix} \hat{y}_{1.1}(t) \\ \hat{y}_{2.1}(t) \end{pmatrix} = \begin{pmatrix} y_1^0 + t y_2^0 \\ y_2^0 + t \{ -y_1^0 + \varepsilon y_2^0 [1 - (y_1^0)^2] + \beta \} \end{pmatrix}$$

or, equivalently, in terms of the original variables

$$\begin{pmatrix} \hat{x}_{1.1}(t) \\ \hat{x}_{2.1}(t) \end{pmatrix} = \begin{pmatrix} x_1^0 + t x_2^0 \\ x_2^0 + t \{ -x_1^0 + \varepsilon x_2^0 [1 - (x_1^0)^2] + \beta \} \end{pmatrix}.$$

Taking into account the relations

$$\begin{cases} \tilde{x}_{1.1}(T) = 0 \\ \tilde{x}_{2.1}(T) = 0 \end{cases} \quad \text{and} \quad \begin{cases} \hat{x}_{1.1}(T) = 0 \\ \hat{x}_{2.1}(T) = 0 \end{cases},$$

we get the following results

$$\tilde{T} = \frac{x_2^0}{x_1^0 - \varepsilon x_2^0 [1 - (x_1^0)^2] - \alpha} > 0 \quad \text{and} \quad \hat{T} = \frac{x_2^0}{x_1^0 - \varepsilon x_2^0 [1 - (x_1^0)^2] - \beta} > 0,$$

with the relations

$$(x_2^0)^2 + (x_1^0)^2 - \varepsilon x_1^0 x_2^0 [1 - (x_1^0)^2] - \alpha x_1^0 = 0 \quad \text{or} \quad (x_2^0)^2 + (x_1^0)^2 - \varepsilon x_1^0 x_2^0 [1 - (x_1^0)^2] - \beta x_1^0 = 0.$$

The optimal time solution to the problem is

$$\min(\tilde{T}, \hat{T}) = \begin{cases} \tilde{T} & \text{if } \frac{x_2^0}{x_1^0 - \varepsilon x_2^0 (1 - (x_1^0)^2) - \alpha} < \frac{x_2^0}{x_1^0 - \varepsilon x_2^0 (1 - (x_1^0)^2) - \beta}, \\ \hat{T} & \text{otherwise.} \end{cases}$$

An integral of motion of the SODE (35) is

$$\begin{aligned} J(x_1, x_2) &= x_2 + \varepsilon \left( \frac{1}{3} x_1^3 - x_1 \right) + \int_0^t [x_1(\tau) - u(\tau)] d\tau \\ J(x_1^0, x_2^0) &= x_2^0 + \varepsilon \left( \frac{1}{3} (x_1^0)^3 - x_1^0 \right). \end{aligned} \quad (41)$$

Setting

(i) for  $u(t) = \alpha$ ,

$$\tilde{\Delta}_1(t) = J(\tilde{x}_{1.1}, \tilde{x}_{2.1}) - J(x_1^0, x_2^0) = \frac{1}{3} \varepsilon (x_2^0)^3 t^3 + \left[ \frac{1}{2} x_2^0 + \varepsilon x_1^0 (x_2^0)^2 \right] t^2$$

and using the Cardan's formulas for cubic polynomials, we obtain

$$|\tilde{\Delta}_1(t)| < \kappa \quad \text{for } t \in \left[ 0, \sqrt[3]{-\frac{q}{2} + \sqrt{Q}} + \sqrt[3]{-\frac{q}{2} - \sqrt{Q}} - \frac{A}{3} \right]$$

with  $\kappa$  a sufficiently small positive number and

$$\begin{aligned} Q &= \left( \frac{p}{3} \right)^3 + \left( \frac{q}{2} \right)^2, \quad p = -\frac{1}{3} A^2, \quad q = \frac{2}{27} A^3 - B, \\ A &= \frac{\frac{3}{2} + 3 |\varepsilon x_1^0 x_2^0|}{|\varepsilon| |x_2^0|^2}, \quad B = \frac{3\kappa}{|\varepsilon| |x_2^0|^3}; \end{aligned}$$

(ii) and for  $u(t) = \beta$ ,

$$\hat{\Delta}_1(t) = \frac{1}{3} \varepsilon (x_2^0)^3 t^3 + \left[ \frac{1}{2} x_2^0 + \varepsilon x_1^0 (x_2^0)^2 \right] t^2;$$

$$\text{we obtain } |\hat{\Delta}_1(t)| < \kappa \text{ for } t \in \left[ 0, \sqrt[3]{-\frac{q_1}{2} + \sqrt{Q_1}} + \sqrt[3]{-\frac{q_1}{2} - \sqrt{Q_1}} - \frac{A_1}{3} \right]$$

with  $\kappa$  a sufficiently small positive number and

$$\begin{aligned} Q_1 &= \left( \frac{p_1}{3} \right)^3 + \left( \frac{q_1}{2} \right)^2, \quad p_1 = -\frac{1}{3} A_1^2, \quad q_1 = \frac{2}{27} A_1^3 - B_1 \\ A_1 &= \frac{\frac{3}{2} + 3 |\varepsilon x_1^0 x_2^0|}{|\varepsilon| |x_2^0|^2}, \quad B_1 = \frac{3\kappa}{|\varepsilon| |x_2^0|^3}. \end{aligned}$$

#### 4. Concluding remarks

In this paper, we have successfully extended the domain of applicability of Bagarello's approach, developed for the analysis of ordinary differential equations, to optimal control problems. The relevance of the derived approximation scheme (18) and error estimation given in formula (19) has been exploited to investigate time-optimal problems of forced harmonic oscillator systems. Three particular cases have been explicitly treated and discussed.

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